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Article *in* PRIMUS: problems, resources, and issues in mathematics undergraduate studies · August 2016

DOI: 10.1080/10511970.2016.1214654

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PRIMUS

Problems, Resources, and Issues in Mathematics Undergraduate Studies

ISSN: 1051-1970 (Print) 1935-4053 (Online) Journal homepage: http://www.tandfonline.com/loi/upri20

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To cite this article: Celil Ekici & Andrew Gard (2017) Inquiry-Based Learning of Transcendental Functions in Calculus, PRIMUS, 27:7, 681-692, DOI: 10.1080/10511970.2016.1214654

To link to this article: https://doi.org/10.1080/10511970.2016.1214654

Accepted author version posted online: 02 Aug 2016.
Published online: 02 Aug 2016.

PRIMUS, 27(7): 681–692, 2017

ISSN: 1051-1970 print / 1935-4053 online
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Inquiry-Based Learning of Transcendental Functions in Calculus

Celil Ekici and Andrew Gard

Abstract: In a series of group activities supplemented with independent explorations and assignments, calculus students investigate functions similar to their own derivatives. Graphical, numerical, and algebraic perspectives are suggested, leading students to develop deep intuition into elementary transcendental functions even as they lay the foundation for understanding infinite series and growth functions later in the course sequence. Students further explore interconnections between such functions, culminating in the synthesis of the Euler formula $e^{ix} = \cos x + i \sin x$. These activities have been implemented over several years with encouraging results.

Keywords: Inquiry-based learning, teaching calculus, transcendental functions, Euler's formula, Taylor series

1. MOTIVATION

Inquiry-based approaches to the teaching of mathematics are known to be appropriate as a matter of cognitive learning theory [2] and effective as a matter of practice [7] in undergraduate mathematics education [15]. Inquiry has been characterized as learning to speak and act mathematically by participating in mathematical discussions, posing conjectures, and solving new or unfamiliar problems [17]. The National Council of Teachers of Mathematics and many others have called for greater implementation of inquiry-based techniques in the learning of mathematics at every level [1, 3, 4, 13].

The project described here is an attempt to address this need at the level of college calculus. In the modules described below, students encounter a wide array of developmentally-appropriate ideas and work to

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synthesize them across algebraic, graphical, and numeric representations in a learning process that is at once self-guided and carefully curated. Ultimately, they emerge with a greater capacity for mathematical investigation as well as deeper understanding of some of the fundamental ideas of calculus.

Our work contributes to a growing body of literature on inquiry-based approaches to transcendental functions. For example, researchers have explored analogies between DNA and sine functions using series approximations [9].

2. SETTING AND APPROACH

The inquiry-based learning (IBL) modules described here have been implemented in several similar formats at the University of the Virgin Islands over the past 3 years, initially in the Calculus I–II series taught by the first author in 2012–13 and 2013–14 and then by the second author the following academic year. During the Fall 2013, while implementing and refining these IBL activities for Calculus I, the first author concurrently attended an online professional development course on Accountable Talk®: Conversation that Works from the University of Pittsburgh's Institute for Learning that was used to support the creation of productive student-centered discourse in a calculus class.

In each case, activities occupied approximately 10 50-minute sessions of instructional time within 3 weeks of the second half of a freshmen calculus course. In our content sequence, students are considered to be ready for these tasks after being introduced to the derivatives of polynomial functions, both with limits and without. Depending on the instructor, our calculus courses are not purely inquiry-based, but incorporate those active learning philosophies throughout. The activities described here can easily be integrated in a traditional calculus setting.

Ours is a constructivist learning approach that focuses on developing conceptual understanding through exploration of a family of mathematical ideas and employing multiple representations in a discussion-oriented learning environment [5, 17, 19]. Drawing from Rasmussen et al.'s inquiry-oriented approaches for students and teachers [16], and the Process Oriented Guided Inquiry Learning (POGIL) strategy [6, 14], we offer an investigative approach to calculus. Working in small groups, students are allocated roles depending on their strengths. If a student is fluent with technological tools for visual representations of the problem, for example, they may lead the group's graphical explorations using software such as GeoGebra. If they are more comfortable in using spreadsheets, then they may lead the numerical activities for the group.

Other key pieces of our pedagogical philosophy include:

- Providing a **supportive** environment for productive mathematical discussion and communication of emerging ideas among peers.
- Supporting **exploration** and **articulation** of patterns across different representations to validate and extend student observations.
- Letting students **disagree** and **convince** each other with viable mathematical arguments and justifications [12].
- Allowing students to **struggle** with problems productively [12].
- Facilitating students' **reflection** and **dialogue** within and across peer groups.
- **Listening to, cultivating, and reinforcing** good talk moves generated by students towards productive mathematical discussions. [5, 11, 16].

Overall, we aim to engage students in social construction of mathematical concepts in an environment that promotes productive mathematical discussions [8, 10, 12, 18].

3. PROJECT DESCRIPTION

3.1. PHASE 1: Finding Polynomials with Similar Derivatives

Shortly after our students learn elementary differentiation techniques, we place them into groups and give them the following prompt: *Try to find a function that looks as much as possible like its own derivative. This similarity can be algebraic, graphical, or numerical*. Since our calculus series follows a late-transcendentals approach, students work only with polynomials and other rational functions in Calculus I.

Working together on this problem, they quickly discover that there are polynomials that agree with their derivatives except for the one term with the highest degree. For example:

$$1, \quad 1+x, \quad 1+x+\frac{1}{2}x^2, \quad 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3, \cdots \tag{1}$$

The emerging polynomials

$$p_n(x) = 1+x+\frac{1}{2}x^2+\cdots+\frac{x^n}{n!} \tag{2}$$

are truncated Taylor series representations of e^x, which students will formally address later in Calculus II. Using the equations above, students generalize the non-matching term as $x^n n!$ for an nth degree polynomial and its derivative.

Students determine the specific flavor of the conversation through discussion and collaboration, so every iteration of the project brings new perspectives. In the fall of 2013, for instance, a group of students started to explore these patterns algebraically by explicitly equating a general third-degree polynomial with its derivative:

$$p(x) = ax^3 + bx^2 + cx + d, \qquad p(x) = 3ax^2 + 2bx + c. \tag{3}$$

After successfully matching three terms this way, they worked on expanding the process to higher-order polynomials. This choice of representation led to polynomials similar to the expansion of e^x.

Also observe that in this case, students initially chose to write their polynomial functions beginning with the terms of highest degree. This is a mathematical habit that eventually needs to be broken as the idea of series representations of functions is developed. Our activities facilitate this transition.

In each implementation, students eventually begin to construct higher-degree polynomials one term at a time. As more terms are added to the emerging series, they notice patterns in the coefficients and naturally are led to investigate the role of recursion in the problem, considering the generalizing term and recognizing and utilizing the factorial idea. Process-building and concept development thus go hand-in-hand.

Building on students' initial algebraic manipulations, we work to flesh out their understanding of the relations between these polynomials with graphical and numerical follow-ups. These activities help our students interpret the local and global behaviors of those polynomials with similar derivatives and better see the relations between them. It is then that we expect to observe an *aha* moment when students finally recognize an explicit relation to exponential functions, with which they are familiar from previous classes.

In a follow-up activity, we ask students to graph and compare polynomials and their derivatives on a common window as seen in Figure 1.

After presenting this task, we step back to listen, wait, and focus the conversation as needed. We expect several of the following questions to emerge and be addressed by students: *What do these graphs have in common? Do any patterns emerge? Where are the graphs very different? Is the difference quantifiable?* The activity gives students both a concrete task and an abstract problem to consider, a combination we have found to be highly productive.

During student group work, we walk around to encourage engagement and to support productive mathematical talk. When we observe students initiating potentially productive discussion threads, but not following up well with the others in the group, we ask them to re-articulate, explain to their peers, and repose their observations and comments. We value student-generated ideas and questions while watching out for emerging content, dialogue, and engagement within groups. At the end of each group session,

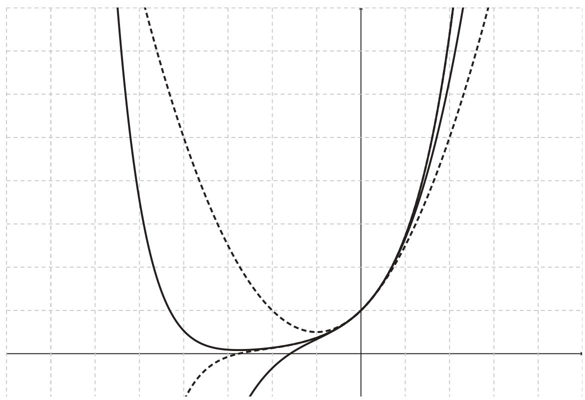

Figure 1. Graphical exploration of polynomials similar to their derivatives. Solid lines represent p_3 and p_8 and the dashed lines represent their derivatives.

we give each group an opportunity to summarize and discuss their findings with the entire class.

Next we prompt students with the following question: *We have seen that these polynomials are algebraically and graphically similar to their derivatives. Are they numerically similar as well?* Students are then asked to work together to build spreadsheets comparing the polynomials of differing degrees at various points near $x = 0$. A typical result is shown in Table 1, where for simplicity only polynomials of degree 3 and degree 8 are shown.

Students use these spreadsheets dynamically, experimenting with different x-values as they explore the numerical match between the various polynomials and their derivatives. They observe that the polynomials have a tighter fit close to zero, and that the intervals on which they match closely get wider as the degree increases. By studying the errors, they are then able to estimate the intervals on which the match is within a given tolerance. As suggested by Table 1, for instance, the third-degree polynomial differs from its derivative by about 0.02 when x-values are within $(-0.5, 0.5)$. Students further tabulate x-values for which the pointwise errors are small for higher-degree polynomials. This interval of close match widens to about $(-2, 2)$ for the eighth-degree polynomial with errors less than 0.01.

In a follow-up assignment, students are asked to verify their numerical observations by solving

$$\left|\frac{x^3}{3!}\right| < 0.02 \quad \text{and} \quad \left|\frac{x^8}{8!}\right| < 0.01. \tag{4}$$

Using such algebraic, graphical, and numerical representations, students come to gain substantial intuition into these polynomials and their apparent relation.

Table 1 Numerical comparison of polynomials of degree 3 and degree 8 with their respective derivatives. The last columns E_3 and E_8 show the differences between these polynomials and their derivatives at the given points

x	p_3	p'_3	p_8	p'_8	E_3	E_8
−2	−0.333	1.000	0.137	0.130	−1.333	0.006
−1	0.333	0.500	0.368	0.368	−0.167	0.000
−0.5	0.604	0.625	0.607	0.607	−0.021	0.000
0	1.000	1.000	1.000	1.000	0.000	0.000
0.5	1.646	1.625	1.649	1.649	0.021	0.000
1	2.667	2.500	2.718	2.718	0.167	0.000
2	6.333	5.000	7.387	7.381	1.333	0.006

Inevitably, the idea of the convergence of sequences of functions comes into play, introduced by the students themselves rather than their instructor.

3.2. PHASE 2: Extending to Other Transcendentals

At this point we slightly alter the original question, suggesting that students explore polynomial functions whose derivatives are the *opposites* of themselves. The initial purpose of this change is to allow students to apply the algebraic, numerical, and graphical strategies they learned in Phase 1 in a different and more challenging context.

Algebraically, students come to recognize that the problem lies essentially in resolving a sign pattern problem in the terms of the polynomials previously generated:

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots. \tag{5}$$

Still working in small groups, they explore how to obtain this alternating series, soon recognizing a connection to the parity of the powers of x in the polynomial and eventually speculating as to the presence of a negative sign in those exponents. Incorporating these new ideas into the polynomial pattern used for the expansion of e^x, students arrive at the series corresponding to e^{-x}. This idea can be reinforced with the same graphical and numerical activities used in Phase 1.

We further extend and explore the notion of alternating sign patterns by asking students to investigate polynomials whose *second* derivatives are the negatives of themselves. Building successive polynomials with more matching terms leads to the discovery of two distinct classes of polynomials that are similar to their second derivatives, depending on whether even- or odd-powered terms are used. For example:

$$C_4 = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \quad \text{and} \quad S_5 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5. \tag{6}$$

We ask students to plot such polynomial functions and discuss their local and global behavior. Finally, we ask them to explain their observations to each other.

As seen in Figure 2, students discover that near zero the even-powered polynomials with alternating signs increasingly behave like the cosine function whereas the others behave like the sine function.

Students develop a consensus around the observation that higher-degree cosine- and sine-like polynomials have wider intervals of local match around zero. We ask students to develop further evidence to support this idea using spreadsheets, comparing the values of cosine and sine to their polynomial counterparts at points near zero.

We encourage students to reflect back on their process and to make explicit the recursive building of polynomials through expressions such as

$$C_{2n}(x) = C_{2n-2}(x) + \frac{(-1)^n x^{2n}}{(2n)!}. \tag{7}$$

This in turn helps them to focus on the role of generating terms in the building of these polynomials.

The second phase ends with an informal discussion of the possibility of using these polynomials to *define* transcendental functions like e^x and $\sin x$. This paves the way for a more rigorous treatment of infinite series in subsequent courses.

3.3. PHASE 3: Synthesis: Euler's Formula

Next is the critical synthesis in which students explore interconnections between series representations of sine, cosine, e^x, and e^{-x}. With the exception

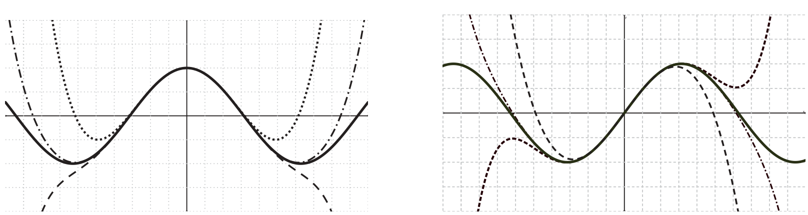

Figure 2. Two classes of polynomials whose second derivatives are negatives of themselves. The graph on the left compares polynomials of degree 4, 6, and 8 with the cosine function. On the right, polynomials of degree 3, 5, and 7 are compared with the sine function.

of the signs, the terms of the sine and cosine polynomials are contained in the polynomial for e^x in its odd- and even-powered terms.

Students investigate how to combine the sine and cosine series to obtain all the terms of e^x and realize that the problem lies in resolving the repeated sign pattern of $\{+, +, -, -\}$ in the coefficients of the combined polynomial.

After initial struggle, some students attempt to connect this problem with the strategy they used in resolving the sign pattern problem for e^{-x} using powers of −1. Building on that idea, we encourage them to look for the mysterious coefficient before x in

$$e^{c \cdot x} = 1 + (cx) + \frac{(cx)^2}{2} + \frac{(cx)^3}{6} + \frac{(cx)^4}{24} + \cdots, \tag{8}$$

needed to produce the sign pattern matching the coefficients for the sine and cosine polynomials. Comparing the signs before the even terms in equation (8) and the cosine polynomial in equation (6), they see that c^2 should be −1 and c^4 should have a value of one. Some students have another *aha* moment, realizing that this number is i. They then verify that $i^0, i^2, \cdots, i^{2n}$ resolve the problem for even-numbered cosine polynomial terms, satisfying the even terms in the sign pattern {+, +, −, −}. They further speculate that this number can even work for the terms of the sine polynomials. Through revoicing and rearticulation, this strategy is tried out in several peer groups.

Students later work together on the verification of the connection between e^{ix} and the sine and cosine polynomials. Summarizing their work, students achieve polynomial representation of *Euler's formula*:

$$e^{ix} = \cos x + i \sin x. \tag{9}$$

We push students to interpret this identity geometrically. Building on their previous knowledge of trigonometry, students re-express the points on a unit circle as (cos x, sin x) in an Argand plane. Euler's formula then brings the geometric perspective where e^{ix} represents rotation of the point (1, 0) around the unit circle by x radians in the counterclockwise direction. In the fall of 2012, two students in their reflection suggested its connections to the unit circle and the idea of rotation. One of these students wrote:

> "the formula describes the movement of a circle. The imaginary exponential growth rotates a number. As a result, $e^{i\pi}$ entails starting at 1 and rotating 180 degrees or π to get to −1. To conclude, the formula is still complex, but not beyond my understanding."

4. REFLECTION AND ASSESSMENT

At the end of the project, students are asked to look back on the process in a final writing assignment and group presentation. This self-reflection gives a window into their developing understanding of the ideas explored and the connections between them. In the fall of 2014, for example, one student wrote, "we graphed the different polynomials and I could see them getting more like e^x." The year before, another student, without being prompted, re-expressed the problem $y' = y$ in the form $\frac{dy}{y} = dx$ to find a solution implicitly, already showing intuition into the deeper connection between the exponential and logarithmic functions.

Students' responses show that they have developed a sense of ownership of the ideas being explored, that they have engaged with the material from multiple perspectives, and that they have expanded their ability to articulate their mathematical thinking in plain language. Student reports and writings work well here to demonstrate their understanding of mathematical connections. Keeping an observation log is helpful to build on some group discussions. We find balanced assessment with a variety of tools, such as written reflections, reports, and exams, fits well with the learning process in an inquiry approach.

5 DISCUSSION AND CONCLUSIONS

These learning modules allow students to become exposed to a wide array of mathematical ideas in a relatively short time span. These include: differential similarity of polynomial functions, factorials, recursion, antiderivatives, approximation of transcendental functions by polynomials, sequence of partial sums, infinite series, Euler's formula as a relationship between the functions e^x, $\cos x$, and $\sin x$, error analysis, and complex numbers in the calculus context. More importantly, students are challenged to explore the *connections* between these ideas, an aspect often overlooked in otherwise strong math classes.

All of these concepts are revisited and formalized in Calculus II even as new ideas are brought into the discussion. Our series approach to trigonometric functions proves to be useful in anchoring numerical techniques for integrating functions such as $\sin(x^2)$. Hyperbolic trigonometric functions also provide a good area for enrichment, as students apply now-familiar algebraic, graphical, and numerical techniques to an important, but new, class of transcendental functions.

Further implementation of these inquiry-based modules will focus on better unification of assignments done outside of class, including online discussion boards and community learning pages, with in-class work. We expect to expand the use of this project, further honing its effectiveness and better quantifying its utility in other calculus settings. We are currently hoping to implement this project in high schools with classroom case studies.

ACKNOWLEDGMENTS

We would like to thank the anonymous reviewers for their insightful comments and suggestions.

FUNDING

We also appreciate support given by the NSF Grant No. 1355437 for the dissemination of these IBL modules for calculus to innovate undergraduate mathematics education.

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BIOGRAPHICAL SKETCHES

Celil Ekici has been working for the University of Virgin Islands since 2012 as an assistant professor of mathematics with a Ph.D. from the University of Georgia. He enjoys integrating different flavors of inquiry into mathematics and education courses for undergraduates and graduates. His research interests involve trigonometric functions in secondary and higher mathematics; mathematical modeling in K-16 STEM education; phenomenology of mathematics; population dynamics.

Andrew Gard joined the University of the Virgin Islands faculty in the fall of 2014 as an assistant professor of mathematics after obtaining his doctorate from the Ohio State University and completing a visiting role at Ohio Wesleyan University. His research interests include geometric optimization and big data analysis. In the classroom, he focuses on piloting and extending the use of inquiry-based techniques at all levels of mathematics.